Functional inequalities on the Gaussian path space

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Quasi-invariant property

- Leb. measure $\lambda(dx)$ on $\mathbb{R}^d$: $x \in \mathbb{R}^d, v \in \mathbb{R}^d$, let

$$\xi_v^\varepsilon(x) := x + \varepsilon v, \quad \varepsilon \in \mathbb{R} \ \Rightarrow \ \frac{d(\xi_v^\varepsilon)_*(\lambda)}{d\lambda} = 1, \quad \Big(\text{Translation invariant}\Big).$$

- Standard Gaussian measure $\gamma(dx)$ on $\mathbb{R}^d$: $x \in \mathbb{R}^d, v \in \mathbb{R}^d$, let

$$\xi_v^\varepsilon(x) := x + \varepsilon v, \quad \varepsilon \in \mathbb{R} \ \Rightarrow \ \frac{d(\xi_v^\varepsilon)_*(\gamma)}{d\gamma} = \exp\left(\varepsilon\langle \cdot, v\rangle - \frac{\varepsilon^2}{2}|v|^2\right).$$

- Wiener Measure μ on $C([0,1]:\mathbb{R}^d)$: let

$$\zeta_h^\varepsilon(\omega) := \omega + \varepsilon h, \quad \omega \in C([0,1]:\mathbb{R}^d), \quad h \in \mathbb{H},$$

where

$$\mathbb{H} := \left\{ h \in C([0,1];\mathbb{R}^d) : h(0) = 0, \|h\|_{\mathbb{H}}^2 := \int_0^1 |\dot{h}_s|^2 ds < \infty \right\}.$$

Then ζ_h^ε for μ is a family of quasi-invariant flow and

$$\frac{d(\zeta_h^\varepsilon)_*\mu}{d\mu} = \exp\left(\varepsilon \int_0^1 \langle h'(s), dB_s\rangle - \frac{\varepsilon^2}{2}\|h\|_{\mathbb{H}}^2\right).$$

- Wiener Measure μ on $C([0,1]:M)$: M Riemannian manifold.

$$\exists\ \zeta_h^\varepsilon : W_x(M) \mapsto W_x(M), \qquad h \in \mathbb{H},$$

s.t.

$$\frac{d(\zeta_h^\varepsilon)_*\mu}{d\mu} = \exp\left[\int_0^\varepsilon \Phi_h^1(\zeta_h^{-\lambda})d\lambda\right],$$

where

$$\Phi_h^t(\gamma) := \int_0^t \left\langle h'(s) + \frac{1}{2}\mathrm{Ric}_{U_s(\gamma)}h(s), dB_s(\gamma)\right\rangle.$$

Driver(92), Hsu(95, 02), Hsu-Ouyang(09), Wang(11), Chen-Li-W(23)

- Fractional Wiener Measure μ on $C([0,1]:\mathbb{R}^d)$, i.e. the law of FBM B^H:

$$B_t^H := \int_0^t K_H(t,s)dB_s, \quad t \geq 0,$$

where $K_H(t,s) =$

$$\begin{cases} c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du, & H \in (\frac{1}{2},1), \\ 1_{[0,t]}(s), & H = \frac{1}{2}, \\ b_H \left(\left(\frac{t(t-s)}{s} \right)^{H-\frac{1}{2}} - \left(H-\frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{1}{2}} u^{H-\frac{3}{2}} du \right), & H \in (0,\frac{1}{2}). \end{cases}$$

Define $\zeta_h^\varepsilon(\omega) := \omega + \varepsilon h, \quad \omega \in C([0,1]:\mathbb{R}^d), \quad h(t) = \int_0^t F(t,s)g(s)ds.$

$$\frac{d(\zeta_h^\varepsilon)_*\mu}{d\mu} = \exp\left(\varepsilon \int_0^1 \langle K_H^{*,-1} K_H^{-1} h'(s), dB_s^H \rangle - \frac{\varepsilon^2}{2} \|h\|_{\mathbb{H}^H}^2\right).$$

Decreusefond-Üstünel(99), Elisa-Mazet-Nualart(01), Fan(15)

Mathematical finance:

- Brownian motion: Theory of option pricing formula(1997, Black-Scholes-Merton)
- Fractional Brownian motion: Describing the price fluctuation behavior model of financial markets
- Gaussian process: Due to the complex financial behaviors of price fluctuations in financial markets, it may not be comprehensive enough to use FBM only to describe its fluctuation behaviors. It is necessary to introduce a more general Gaussian process.
- Greek alphabet for the derivative

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbb{E}(\Phi(B)(\varepsilon)) &= \mathbb{E}\Big(\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Phi(B)(\varepsilon)\Big)\\ &= \mathbb{E}\Big(\Phi(B)(\varepsilon)\alpha_\varepsilon(\Phi(B))\Big) ------\text{Bismut's Formula}.\end{aligned}$$

Our goal

- F-Gaussian process:

$$B_t^F := \int_0^t F(t,s)dB_s, \quad t \geq 0.$$

- Quasi-invariant theorem

- Functional inequality

F-Gaussian process

Let $F \in C^2(\mathbb{R} \times \mathbb{R})$. Consider

$$B_t^F := \int_0^t F(t,s)\mathrm{d}B_s, \quad t \geq 0.$$

In this talk, we always assume that

- $F \in L^2(\mathbb{R}^2)$ and $\int_0^{t\wedge s} F(t,r)F(s,r)\mathrm{d}r \neq 0, \quad \forall\ t,s \in \mathbb{R}$
- $\varphi_t(\cdot) := F(t,\cdot) \in L^2(\mathbb{R}), \quad \forall t \geq 0.$

This implies that B^F is a centered Gaussian process with covariance

$$\mathbb{E}(B_t^F B_s^F) := R_F(t,s) = \int_0^{t\wedge s} F(t,r)F(s,r)dr.$$

According to the definition of B^F:

$$B_t^F := \int_0^t F(t,s)\mathrm{d}B_s, \quad t \geq 0,$$

the kernel function F is the key point between B and B^F. Here we introduce the operator $K_F : L^2([0,T];\mathbb{R}) \to L^2([0,T];\mathbb{R})$ defined by

$$K_F h(t) := \int_0^t F(t,s)h(s)ds, \quad h \in L^2([0,T]^2;\mathbb{R}).$$

To characterize the connection between B^F and B, we need to introduce K_F^* and $(K_F^*)^{-1}$ operators.

$$(K_F^* h)(s) := F(s,s)g(s) + \int_s^T \frac{\partial F(t,s)}{\partial t} h(t)dt, \quad h \in L^2([0,T]^2;\mathbb{R}).$$

This requires that

$$(A) \quad a.s.\ t \in L^2[0,T], \qquad F_1(t,s) := \frac{\partial F(t,s)}{\partial t} \in L^2([0,T]^2;\mathbb{R})$$

To make sure that $(K_F^*)^{-1}: L^2([0,T];\mathbb{R}) \to L^2([0,T];\mathbb{R})$ is bounded operator. we need to assume that (B):

$\exists\ \Theta \in L^2([0,T]^2) \cap C^1((0,T)^2)$ and $0 < \zeta_1, \zeta_2 \in L^2([0,T];\mathbb{R})$ with $\frac{1}{\zeta_1}, \frac{1}{\zeta_2} \in L^2([0,T];\mathbb{R})$ s.t.

$$\begin{cases} \frac{\Theta(r,t)}{\zeta_1(r)} F(r,r) + \int_t^r \frac{\Theta(s,t)}{\zeta_1(s)} \frac{\partial F(r,s)}{\partial r} ds = \zeta_2(r), \\ \frac{\Theta(r,t)}{\zeta_2(t)} F(t,t) + \int_t^r \frac{\partial F(s,t)}{\partial s} \frac{\Theta(r,s)}{\zeta_2(s)} ds = \zeta_1(t), \qquad \forall\ 0 \le t \le r \le T. \\ \frac{\partial \Theta(r,t)}{\partial r} + \frac{\partial \Theta(r,t)}{\partial t} \equiv 0. \end{cases}$$

Example: (1) Let

$$F(t,s) = \frac{c}{\Gamma(\alpha)} f_1(s) \int_s^t (u-s)^{\alpha-1} f_2(u) du, \quad 0 < \alpha \le 1.$$

In particular, if

$$f_1(s) = s^{\frac{1}{2}-H}, \ \ f_2(t) = t^{H-\frac{1}{2}}, \ \ \alpha = H - \frac{1}{2}, \ \ c = c_H \Gamma\Big(H - \frac{1}{2}\Big).$$

Then $F(t,s) = K_H(t,s)$.

(2) Let $F(t,s) = f(s)$ for some f.

Remark

(1) *When*

$$F(t,s)=\frac{c}{\Gamma(\alpha)}f_1(s)\int_s^t (u-s)^{\alpha-1}f_2(u)du, \quad 0<\alpha\le 1.$$

Let

$$\begin{cases} \Theta(t,s)=\frac{1}{\Gamma(1-\alpha)}(t-s)^{-\alpha}, & s\le t \\ \zeta_1(t)=f_1(t), & t\in[0,T] \\ \zeta_2(t)=f_2(t), & t\in[0,T]. \end{cases}$$

(2) *When* $F(t,s)=f(s)$ *for some* f. *Let*

$$\Theta(t,s)=1, \;\; \zeta_1\equiv 1, \quad \zeta_2=f(t).$$

Existence of $(K_F^*)^{-1}$

Theorem

Suppose that (A) and (B). Then

$$(K_F^*)^{-1} h(t) = -\frac{1}{\zeta_2(t)} \frac{d}{dt} \int_t^T \frac{\Psi(s,t)}{\zeta_1(s)} h(s) ds, \quad h \in L^2([0,T];\mathbb{R}).$$

The Cameron-Martin space $\mathbb{H}^F$ for B^F is denoted by

$$\mathbb{H}^F := \left\{ h^F := K_F \dot{h} \middle| h \in \mathbb{H}, \|h^F\|_{\mathbb{H}^F}^2 < \infty \right\},$$

where

$$\langle h^F, g^F \rangle_{\mathbb{H}^F} = \int_0^T h'(s) g'(s) ds, \quad h^F, g^F \in \mathbb{H}^F.$$

For each fixed $T > 0$, let $\mathbf{H}^F := (K_F^*)^{-1}\big(L^2([0,T];\mathbb{R})\big)$. Let $\psi \in \mathbf{H}^F$ and Δ be a partition of the interval $[0,T]$:

$$\Delta = \{0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T\}.$$

Integrals for F-Gaussian process

Define the B^F-integral with respect to ψ:

$$\int_0^T \psi(s)dB_s^F := \lim_{|\Delta|\to 0} \sum_{i=0}^{n-1} \left(\frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} \psi(s)ds \right) \left(B_{t_{i+1}}^F - B_{t_i}^F \right).$$

Theorem

Suppose that (A) and (B). Then

$$\int_0^T \psi(s)dB_s^F = \int_0^T K_F^* \psi(s)dB_s, \quad \forall \psi \in \mathbf{H}^F.$$

Neven(68), Decreusefond-Ustunel(99), Duncan-Hu-Pasik-Duncan(00), Nualart(03)

Quasi-invariant Theorem

Let

$$R_F := K_F \circ K_F^*.$$

Theorem (Chen-Sun-W 23+)

Assume that $F(t,s) = f(t)\hat{F}(t,s), \quad t \geq s \geq 0$, *where* $\hat{F}$ *satisfies conditions (A) and (B), and for some function* f *with* $f, \frac{1}{f} \in L^2([0,T];\mathbb{R})$. *Then for each* $h^F \in \mathbb{H}^F$, *we have*

$$\int_\Omega G(B^F(\omega) + h^F) d\mathbb{P}(\omega) = \int_\Omega G(B^F(\omega)) \alpha_{h^F}(\omega) d\mathbb{P}(\omega)$$

for any bounded Borel measurable function G, *where*

$$\alpha_{h^F} = \exp\left\{ \int_0^T (R_{\hat{F}})^{-1} \frac{h^F}{f} dB^{\hat{F}} - \frac{1}{2} \left\| \frac{h^F}{f} \right\|_{\mathbb{H}^{\hat{F}}}^2 \right\}.$$

Integration by parts formula

Define

$$\mathcal{F}C_b = \Big\{ G(\gamma) = g\Big(\int_0^T g_1(s,\gamma_s)\mathrm{d}s, \dots, \int_0^T g_m(s,\gamma_s)\mathrm{d}s \Big) : \quad \gamma \in W_T,$$
$$g \in Lip_b(\mathbb{R}^m), g_i \in C^{0,1}([0,T]\times\mathbb{R};\mathbb{R}), 1 \le i \le m \Big\}.$$

Theorem (Chen-Sun-W 23+)

Assume that $F(t,s) = f(t)\hat{F}(t,s) \quad t \ge s \ge 0$, where $\hat{F}$ satisfies conditions (A) and (B), and for some function f with $f, \frac{1}{f} \in L^2([0,T];\mathbb{R})$. For each $h^F \in \mathbb{H}^F$ and $G_1, G_2 \in \mathcal{F}C_b$, we have

$$\int_\Omega G_2 D_{h^F} G_1 d\mu^F = \int_\Omega G_1 D^*_{h^F} G_2 d\mu^F,$$

where

$$D^*_{h^F} G_2 = -D_{h^F} G_2 + G_2 \int_0^T (R_{\hat{F}})^{-1} \frac{h^F(t)}{f(t)} dB_t^{\hat{F}}.$$

Martingales with respect to B^F

In general, B^F is not a martingale. Does there exist a function $\Psi(t, s)$ such that

$$M_t = \int_0^t \Psi(t, s) \mathrm{d}B^F$$

is a martingale?

Theorem (Chen-Sun-W 23+)

Assume that F satisfies conditions (A) and (B). Then then stochastic process

$$M(t) = \int_0^t \frac{\Theta(t, s)}{\zeta_1(s)\zeta_2(s)} dB_s^F \tag{1}$$

is a martingale with respect to the filtration generated by the F-Gaussian process, where Θ, ζ_1 and ζ_2 are from the condition (B).

Bismut-Elworthy-Li's formula

For each $x \in \mathbb{R}$, define

$$B_t^{F,x+\varepsilon} = B_t^F + x.$$

Theorem (Chen-Sun-W 23+)

Assume that F satisfies conditions (A) and (B). Then for each $T > 0$ and $g \in \mathcal{B}_b(\mathbb{R})$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{E}\left(g(B_T^{F,x})\right) = \frac{1}{T_0}\mathbb{E}\left(g(B_T^{F,x})\left(\int_0^T (R_F)^{-1}\left(\int_0^t F(t,s)\mathrm{d}s\right)\mathrm{d}B_t^{F,x}\right)\right),$$

where

$$T_0 := \int_0^T F(T,s)\mathrm{d}s.$$

Bismut(84), Elworthy-Li(94), Fan(15)

Functional inequality

Quasi-regular Dirichlet form:

For any $G \in \mathcal{F}C_b$, define

- Damped gradient: $D_h G = \langle D^F G, h\rangle_{\mathbb{H}^F}, \quad \forall h \in \mathbb{H}^F$
- Malliavin gradient: $D_h G = \langle DG, h\rangle_{\mathbb{H}}, \quad \forall h \in \mathbb{H}^F$
- L^2-gradient: $D_\varphi G(\omega) = \langle \nabla G(\omega), \varphi\rangle_{L^2([0,T];\mathbb{R})}, \quad \varphi \in L^2([0,T];\mathbb{R})$.

Define by the quadratic forms

$$\mathcal{E}(G,G) := \int_{W_T} \|DG\|^2_{\mathbb{H}^F} \mathrm{d}\mu^F, \quad G \in \mathcal{F}C_b,$$

$$\mathcal{E}_{OU}(G,G) := \int_{W_T} \|DG\|^2_{\mathbb{H}} \mathrm{d}\mu^F, \quad G \in \mathcal{F}C_b,$$

$$\mathcal{E}_{L^2}(G,G) := \int_{W_T} \|\nabla G\|^2_{L^2} \mathrm{d}\mu^F, \quad G \in \mathcal{F}C_b.$$

Quasi-regular Dirichlet form

Theorem (Chen-Sun-W 23+)

Assume that $F(t,s) = f(t)\hat{F}(t,s) \quad t \geq s \geq 0$, *where* $\hat{F}$ *satisfies conditions (A) and (B), and for some function* f *with* $f, \frac{1}{f} \in L^2([0,T];\mathbb{R})$. *Then*$(\mathcal{E}, \mathcal{F}C_b), (\mathcal{E}_{OU}, \mathcal{F}C_b)$ *and* $(\mathcal{E}_{L^2}, \mathcal{F}C_b)$ *are closable and their closures are quasi-regular Dirichlet forms.*

Remark

- *O-U case: Driver-Röckner(92), Löbus(04), Wang-W(08), Wang-W(09), Chen-W(14), Wang(book)*
- L^2*-case: Röckner-W-Zhu-Zhu(20), Chen-W-Zhu-Zhu(21)*

Damped Logarithmic Sobolev inequality

Theorem (Clark-Ocone-Haussmann formula)

Assume that F satisfies conditions (A) and (B). Then for each $G \in \mathcal{F}C_b^\infty$ we have

$$G = \mathbb{E}(G) + \int_0^T H_t^G dB_t^F,$$

where

$$H_t^G = (K_F^*)^{-1}\mathbb{E}(K_F^{-1} D_t^F G | \mathcal{F}_t^F).$$

Fang(CRASPS 94); Decreusefond-Üstünel(PA 99)

Theorem (Chen-Sun-W 23+)

Assume that F satisfies conditions (A) and (B). Then we have

$$\mathbf{Ent}_{\mu^F}(G^2) \le 2\mathcal{E}(G, G) \tag{2}$$

for $G \in \mathcal{F}C_b$.

Logarithmic Sobolev inequality(O-U D-f)

Corollary (Chen-Sun-W 23+)

Assume that F satisfies conditions (A) and (B). Then we have

$$\mathbf{Ent}_{\mu^F}(G^2) \leq C\mathcal{E}_{OU}(G, G),$$

for $G \in \mathcal{D}(\mathcal{E}_{OU})$, where

$$C = 2 \sup_{t\in[0,T]} F(t,t) + 2 \left\| \frac{\partial F(t,s)}{\partial t} \right\|_{L^2([0,T]^2;\mathbb{R})}.$$

(Fang 94), (Hsu 97), (Hino 98), (Wang 02), (Fang-Wang 04), (Fang-Shao 05), (Elworthy-LeJan-Li 07), (Chen-W 14)

Logarithmic Sobolev inequality(L^2-D-f)

Corollary (Chen-Sun-W 23+)

Assume that F satisfies conditions (A) and (B). Then we have

$$\mathbf{Ent}_{\mu^F}(G^2) \leq C\mathcal{E}_{L^2}(G,G), \quad F \in \mathcal{D}(\mathcal{E}_{L^2}), \tag{3}$$

where

$$C = 2\|F\|^2_{L^2([0,T]^2;\mathbb{R})} = 2\int_0^T \int_0^T F(t,s)^2 \mathrm{d}s\mathrm{d}t. \tag{4}$$

Fang(94), Gourcy-Wu(06), Rockner-W-Zhu-Zhu(20), Chen-W-Zhu-Zhu(21), Decreusefond-Üstünel(99), Fan(15)

Differential Harnack inequality

Theorem (W. 23+)

$$\frac{\mathbb{E}\left[D_h D_h G\right]}{\mathbb{E}\left[G\right]} - \frac{\left[\mathbb{E}(D_h G)\right]^2}{\mathbb{E}\left[G\right]^2} + \frac{1}{2}\left\|h^F\right\|_{\mathbb{H}^F}^2 \geq 0, \quad h \in \mathbb{H}^F, G \in \mathcal{F}C_b.$$

Transport cost inequality

Let $W = C([0,1];\mathbb{R})$. Define the Cameron-Martin distance on W:

$$d_{\mathbb{H}^F}(\omega_1,\omega_2) := \begin{cases} \|\omega_1-\omega_2\|_{\mathbb{H}^F}, & \omega_1-\omega_2 \in \mathbb{H}^F, \\ +\infty, & \textit{otherwise}. \end{cases}$$

Define the L^2-Wasserstein distance:

$$W^2_{d_{\mathbb{H}^F}}(G\mu^F,\mu^F) := \inf_{\pi\in\mathcal{C}(G\mu^F,\mu^F)} \left\{ \int_{W\times W} d^2_{\mathbb{H}^F}(\omega_1,\omega_2)d\pi(\omega_1,\omega_2) \right\}.$$

TCI for Cameron-Martin distance

Theorem (W 23+)

Assume that F satisfies conditions (A) and (B). Then

$$W^2_{d^F_{\mathbb{H}}}(G\mu^F, \mu^F) \leq 2\mu^F(G\log G), \quad G \geq 0, \mu^F(G) = 1.$$

TCI for d_∞ and L^2

Let $W = C([0,1];\mathbb{R})$. Define two distances on W:

$$d_\infty(\omega_1,\omega_2) := sup_{0\le t\le 1}|\omega_1(t)-\omega_1(t)|,$$

$$d_{L^2}(\omega_1,\omega_2) := \left\{ \int_0^1 |\omega_1(t)-\omega_1(t)|^2 dt \right\}^{1/2}.$$

Define the L^2-Wasserstein distance:

$$W_d^2(G\mu^F,\mu^F) := \inf_{\pi\in\mathcal{C}(G\mu^F,\mu^F)} \left\{ \int_{W\times W} d^2(\omega_1,\omega_2)d\pi(\omega_1,\omega_2) \right\}.$$

Theorem (W 23+)

Assume that F satisfies conditions (A) and (B). Then $\exists$ $C_1(F)$ *and* $C_2(F)$ *s.t.*

$$W^2_{d_\infty}(G\mu^F, \mu^F) \le C_1(F)\mu^F(G \log G), \quad G \ge 0, \mu^F(G) = 1,$$

and

$$W^2_{d_{L^2}}(G\mu^F, \mu^F) \le C_2(F)\mu^F(G \log G), \quad G \ge 0, \mu^F(G) = 1.$$

- Talagrand(96), Otto-Villani(00), Bobkov-Gentil-Ledoux(01) $\cdots\cdots$
- Feyel-Üstünel(02), Djellout-Guillin-Wu(04), Wang(04), Fang-Shao(05), Saussereau(12), Riedel(17) $\cdots$

Thank you!